HOLOMORPHIC HARMONIC ANALYSIS ON COMPLEX REDUCTIVE GROUPS

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Abstract. We define the holomorphic Fourier transform of holomorphic functions on complex reductive groups, prove some properties like the Fourier inversion formula, and give some applications. The definition of the holomorphic Fourier transform makes use of the notion of K-admissible measures. We prove that K-admissible measures are abundant, and the definition of holomorphic Fourier transform is independent of the choice of K-admissible measures.

1. Introduction

Let f be a holomorphic function on $\mathbb{C} \setminus \{0\}$. It is a standard result in complex analysis that the Laurant series

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n$$

of f converges locally uniformly on $\mathbb{C} \setminus \{0\}$. From the viewpoint of representation theory, $z \mapsto z^n$ ($n \in \mathbb{Z}$) are the holomorphic representations of the multiplicative group $\mathbb{C} \setminus \{0\}$. In this paper, we generalize this fact to complex reductive groups.

Let G be a complex reductive group. Denote the space of holomorphic functions on G by $\mathcal{H}(G)$. Our main goal is to expand any $f \in \mathcal{H}(G)$ as a holomorphic Fourier series

$$f = \sum_{\pi,i,j} \lambda_{\pi,ij} \pi_{ij},$$

which converges locally uniformly on G, where π runs over all holomorphic representations of G, π_{ij} are the matrix elements of π. Note that $\mathcal{H}(G)$ can be endowed with a structure of Fréchet space for which convergence is equivalent to locally uniform convergence, the Fourier expansion implies that the subspace $\mathcal{E}$ of $\mathcal{H}(G)$ consists of linear combinations of matrix elements of holomorphic representations of G is dense in $\mathcal{H}(G)$. We then prove that the holomorphic Fourier expansion satisfies the usual properties of Fourier expansion like the Fourier inversion formula and the Plancherel Theorem. We also provide applications of the holomorphic Fourier expansion to holomorphic class functions and holomorphic evolution partial differential equations on G.

To do this, we select a class of auxiliary measures on G, which are called *K-admissible measures* in this paper. A measure $d\mu$ on G is K-admissible if

(1) $d\mu$ is of the form $d\mu(g) = \mu(g)dg$, where dg is a (left) Haar measure, μ is a

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measurable function on G locally bounded from below;
(2) all holomorphic representations of G are L^2-$d\mu$-integrable; and
(3) $d\mu$ is K-bi-invariant, where K is a chosen maximal compact subgroup of G.
Condition (1) implies that an L^2-$d\mu$-convergent sequence of holomorphic functions is locally uniformly convergent. So we can prove the locally uniform convergence of a sequence of holomorphic functions by showing that it is L^2-$d\mu$-convergent, which is usually easier to handle. Condition (2) ensures that the matrix elements of holomorphic representations are L^2-$d\mu$-integrable. So we can first expand functions as Fourier series in the L^2 sense, and get the locally uniform convergence by Condition (1). Condition (3) enable us to use representation theory of the compact group K. We will show that K-admissible measures on G are abundant enough. In particular, for any $f \in \mathcal{H}(G)$, there exist a K-admissible measure $d\mu$ such that f is L^2-$d\mu$-integrable.

The main idea to expand holomorphic functions on G as Fourier series is as follows. We first prove two theorems of Peter-Weyl-type on G, which, among other things, provide an orthogonal basis of the Hilbert space of L^2-$d\mu$-integrable holomorphic functions, consisting of the matrix elements of holomorphic representations of G. Then, for a holomorphic function f on G that is to be expanded, we choose a K-admissible measure $d\mu$ such that f is L^2-$d\mu$-integrable. By the Peter-Weyl-type theorem, f can be expanded as a Fourier series in the L^2 sense. Since L^2-$d\mu$-convergence implies locally uniform convergence, we get the locally uniformly convergent Fourier expansion of f. Then we prove that such expansion is independent of the choice of the K-admissible measure $d\mu$.

Now we review the contents of the following sections more closely. After a brief recollection of properties of complex reductive groups in Section 2, we will define the notion of K-admissible measures in Section 3, and prove that they are abundant. Two theorems of Peter-Weyl-type, that is, the L^2 case and the locally uniform case, will be proved in Section 4. The L^2 Peter-Weyl-type theorem is due to Hall [1]. In his proof of the completeness part of the theorem, Hall used some analytical techniques like change-of-variables on G, Laplace operators, and the Monotone Convergence Theorem. Our proof of the completeness part will be group-representation-theoretic. We will make use of the complete reducibility of representations of compact groups. The holomorphic Fourier transform will be studied in Section 5. The Fourier inversion formula and the Plancherel Theorem will be proved. Some basic properties of the holomorphic Fourier transform will also be given. Section 6 will be devoted to applications of the holomorphic Fourier transform to holomorphic class functions and holomorphic evolution partial differential equations on complex reductive groups.

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2. Preliminaries on complex reductive groups

The notion of "reductive groups" has different definitions by different authors. For technical reason, we adopt Hochschild [2]. We briefly recall some properties of complex reductive groups below, which are to be used in the following sections. The detailed proofs can be found in [2] or other textbooks on Lie groups. For simplicity, all Lie groups in this paper are assumed to be connected, although many assertions also hold if "connected" is replaced by "with finitely many connected components".

Definition 2.1. (Hochschild [2]) A complex Lie group G is *reductive* if G admits a faithful finite-dimensional holomorphic representation and every finite-dimensional holomorphic representation of G is completely reducible.

Every complex semisimple Lie group is reductive. $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$, as a multiplicative group, is reductive. But $\mathbb{C}$ and $\mathbb{C}/\mathbb{Z}^2$, with the canonical holomorphic structures, are not reductive. In fact, we have

Proposition 2.1. *A complex Lie group G is reductive if and only if G has the form $(H \times (\mathbb{C}^*)^n)/\Gamma$, where H is a simply connected complex semisimple Lie group, Γ is a finite central subgroup of $H \times (\mathbb{C}^*)^n$.*

The following two propositions concern the relations of complex reductive groups and their maximal compact subgroups. For a compact Lie group K, we denote its complexification by $K_{\mathbb{C}}$.

Proposition 2.2. *The complexification of a compact Lie group is reductive. Conversely, all maximal compact subgroups of a complex reductive group is connected, and any two of them are conjugate. Moreover, if K is a maximal compact subgroup of a complex reductive group G, then $G \cong K_{\mathbb{C}}$.*

Proposition 2.3. *Let G be a complex reductive group, K a maximal compact subgroup of it. Then every finite-dimensional unitary representation $\sigma : K \to U(n)$ of K can be uniquely extended to a holomorphic representation $\sigma_{\mathcal{H}} : G \to GL(n, \mathbb{C})$, and σ is irreducible if and only if $\sigma_{\mathcal{H}}$ is irreducible.*

We denote by $\widehat{K}$ the unitary dual of K, and by $\widehat{G}_{\mathcal{H}}$ the set of equivalence classes of all finite-dimensional holomorphic irreducible representations of G. By the above proposition, we have a bijection $\widehat{G}_{\mathcal{H}} \leftrightarrow \widehat{K}$ ($[\pi] \leftrightarrow [\pi|_K]$), where $[\cdot]$ denote the equivalence class of a representation.

The next proposition will be useful in the following sections.

Proposition 2.4. *([4], Lemma 4.11.13) Let G be a complex reductive group, K a maximal compact subgroup of it. Suppose f_1, f_2 are holomorphic functions on G. If $f_1(x) = f_2(x)$ for all $x \in K$, then $f_1 = f_2$ on G.*

3. K-admissible measures

In this section we introduce the notion of K-admissible measures on complex reductive groups, and prove that K-admissible measures are abundant enough. To do this, we first consider a class of measures on general Lie groups.

Definition 3.1. Let G be a Lie group. A measure $d\mu$ on G is *tame* if it has the form $d\mu = \mu dg$, where dg is a (left) Haar measure on G, μ is a measurable function on G which is locally bounded from below, in the sense that, for every $x \in G$, there is a $\delta > 0$ and a neighborhood U of x such that $\mu(y) \geq \delta$ for almost all $y \in U$ (with respect to dg).

Let G be a complex Lie group. Denote the space of continuous functions on G by $C(G)$. For $g \in G$, we define the *right action* R_g and the *left action* L_g of g on $C(G)$ by

$$(R_g f)(h) = f(hg),$$
$$(L_g f)(h) = f(g^{-1}h),$$

where $f \in C(G), h \in G$.

Lemma 3.1. *Let G be a Lie group. Suppose $\mathcal{F}$ is a countable subset of $C(G)$. Then there exists a tame measure $d\mu$ on G such that for all $g_1, g_2 \in G, f \in \mathcal{F}$, we have $R_{g_1} L_{g_2} f \in L^2(G, d\mu)$.*

Proof. Let $\mathcal{F} = \{f_1, f_2, \cdots\}$. Choose a left-invariant metric $d(\cdot,\cdot)$ on G. Then for $g, h \in G$, we have

$$d(e, gh) \leq d(e, g) + d(g, gh) = d(e, g) + d(e, h). \tag{3.1}$$

Let

$$K_n = \{g \in G : d(e, g) \leq n\}$$

for each $n \in \mathbb{N}$, and denote

$$M_n = \max\{|f_k(g)|^2 : g \in K_n, 1 \leq k \leq n\}.$$

For each $n \in \mathbb{N}$, choose $a_n > 0$ such that

$$a_n M_{2n} |K_n \backslash K_{n-1}| \leq \frac{1}{2^n},$$

where $|S|$ denotes the measure of a subset $S \subset G$ with respect to a fixed Haar measure dg on G. We may also assume that $a_{n+1} \leq a_n$ for each n. Define $\mu(g) = a_n$ when $g \in K_n \backslash K_{n-1}$, and let $d\mu(g) = \mu(g)dg$. It is obvious that $d\mu$ is a tame measure. We claim that $d\mu$ satisfies the conclusion of the lemma. In fact, for any pair $h_1, h_2 \in G$, choose a positive integer N such that $\max\{d(e, h_1), d(e, h_2^{-1})\} \leq \frac{N}{2}$. By the inequality (3.1), we have

$$d(e, h_2^{-1} g h_1) \leq d(e, g) + N.$$

This means that $h_2^{-1} K_n h_1 \subset K_{n+N}$ for each $n \in \mathbb{N}$. Now for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\int_G |(R_{h_1} L_{h_2} f_n)(g)|^2 \, d\mu(g) \\
=& \int_{K_{N-1}} |f_n(h_2^{-1} g h_1)|^2 \mu(g) \, dg + \sum_{n=N}^{+\infty} \int_{K_n \backslash K_{n-1}} |f_n(h_2^{-1} g h_1)|^2 \mu(g) \, dg \\
\leq& \int_{K_{N-1}} M_{2N-1} a_1 \, dg + \sum_{n=N}^{+\infty} \int_{K_n \backslash K_{n-1}} M_{n+N} a_n \, dg \\
=& M_{2N-1} a_1 |K_{N-1}| + \sum_{n=N}^{+\infty} M_{n+N} a_n |K_n \backslash K_{n-1}| \\
\leq& M_{2N-1} a_1 |K_{N-1}| + \sum_{n=N}^{+\infty} M_{2n} a_n |K_n \backslash K_{n-1}| \\
\leq& M_{2N-1} a_1 |K_{N-1}| + \sum_{n=N}^{+\infty} \frac{1}{2^n} \\
<& +\infty.
\end{aligned}$$

So $R_{h_1} L_{h_2} f_n \in \mathcal{H}L^2(G, d\mu)$. This proves the lemma. □

Now let G be a complex reductive group. Let $\mathcal{H}(G)$ denote the Fréchet space of holomorphic functions on G, with respect to the semi-norms $p_X(f) = \sup_{g \in X} |f(g)|$, where X is any compact subset of G. Convergence of a sequence in $\mathcal{H}(G)$ with respect to the Fréchet topology is equivalent to locally uniform convergence. For

a tame measure $d\mu$ on G, let $\mathcal{H}L^2(G,d\mu)$ denote the space of L^2-$d\mu$-integrable functions in $\mathcal{H}(G)$, with the inner product

$$\langle f_1, f_2\rangle = \int_G f_1(g)\overline{f_2(g)}\,d\mu(g).$$

If a sequence $f_1, f_2, \cdots$ in $\mathcal{H}L^2(G,d\mu)$ L^2-converges to a measurable function f, then it also converges locally uniformly to f. This implies that $f \in \mathcal{H}L^2(G,d\mu)$. Hence $\mathcal{H}L^2(G,d\mu)$ is a Hilbert space.

Let K be a maximal compact subgroup of G. For a finite-dimensional holomorphic representation π of G with representation space V_π of dimension d_π, we always choose a fixed inner product $\langle\cdot,\cdot\rangle$ on V_π such that the restriction of π to K is unitary. We also choose a fixed orthonormal basis $\{e_j\}$ of V_π with respect to this inner product, and let

$$\pi_{ij}(g) = \langle \pi(g)e_j, e_i\rangle$$

be the matrix elements. For an $n\times n$ matrix A, we let $\|A\|^2 = \mathrm{tr}(AA^*) = \sum_{i,j=1}^n |A_{ij}|^2$. Then we define

$$C_{\pi,\mu} = \int_G \|\pi(g)\|^2\,d\mu(g)$$

for a tame measure $d\mu$ on G.

Definition 3.2. Let the notations be as above. A tame measure $d\mu$ on G is K-*admissible* if it is K-bi-invariant and such that $C_{\pi,\mu} < +\infty$ for each $[\pi] \in \widehat{G}_{\mathcal{H}}$.

Theorem 3.2. *Let G be a complex reductive group, $\mathcal{F}$ be a countable subset of $\mathcal{H}(G)$. Then for any maximal compact subgroup K of G, there exists a K-admissible measure $d\mu$ such that for all $g_1, g_2 \in G, f \in \mathcal{F}$, we have $R_{g_1}L_{g_2}f \in \mathcal{H}L^2(G,d\mu)$. In particular, we have $\mathcal{F} \subset \mathcal{H}L^2(G,d\mu)$.*

Proof. Applying Lemma 3.1 to the countable set of holomorphic functions $\mathcal{F}' = \mathcal{F}\bigcup\{\pi_{ij} : i,j = 1,\cdots,d_\pi, [\pi] \in \widehat{G}_{\mathcal{H}}\}$, we get a tame measure $d\nu$ on G such that $R_{g_1}L_{g_2}\pi_{ij}, R_{g_1}L_{g_2}f \in \mathcal{H}L^2(G,d\nu)$ for all $g_1, g_2 \in G, f \in \mathcal{F}$. Suppose $d\nu(g) = \nu(g)dg$, and let

$$\mu(g) = \int_K\int_K \nu(xgy)\,dxdy,$$

where dx, dy refer to the Haar measure on K. Then $\mu(g)$ is a K-bi-invariant function on G locally bounded from below, and then the measure $d\mu(g) = \mu(g)dg$ is a K-bi-invariant tame measure. We claim that if a function $f \in \mathcal{H}(G)$ such that $R_{x_1}L_{x_2}f \in \mathcal{H}L^2(G,d\nu)$ for all $x_1, x_2 \in K$, then $f \in \mathcal{H}L^2(G,d\mu)$. In fact,

$$\begin{aligned}
&\int_G |f(g)|^2\,d\mu(g)\\
=&\int_G |f(g)|^2\mu(g)\,dg\\
=&\int_G\int_K\int_K |f(g)|^2\nu(xgy)\,dxdydg\\
=&\int_K\int_K\left(\int_G |f(x^{-1}gy^{-1})|^2\nu(g)\,dg\right)dxdy\\
=&\int_K\int_K\left(\int_G |(R_{y^{-1}}L_xf)(g)|^2\,d\nu(g)\right)dxdy.
\end{aligned}$$

By the assumption, $\int_G |(R_{y^{-1}}L_x f)(g)|^2\, d\nu(g) < +\infty$, and a standard analytical argument shows that it is continuous in (x, y). So $\int_G |f(g)|^2\, d\mu(g) < +\infty$, that is, $f \in \mathcal{H}L^2(G, d\mu)$. Applying this fact to the functions $\pi_{ij}, R_{g_1}L_{g_2}f$ $(f \in \mathcal{F})$, we get $\pi_{ij} \in \mathcal{H}L^2(G, d\mu)$ (which means that $C_{\pi,\mu} < +\infty$) and $R_{g_1}L_{g_2}f \in \mathcal{H}L^2(G, d\mu)$. So the measure $d\mu$ is K-admissible and satisfies the conclusion of the theorem. □

4. Theorems of Peter-Weyl-type

The classical Peter-Weyl Theorems claim that the linear span of matrix elements of irreducible representations of a compact group is uniformly dense in the Banach space continuous functions on the group, and is L^2-dense in the Hilbert space of L^2-integrable functions, which are the starting point of harmonic analysis on compact groups. We prove in this section the similar results for complex reductive groups, which are the base of holomorphic harmonic analysis on such groups. The L^2 analog is due to Hall [1], Theorems 9 and 10. But our proof of the completeness part of the L^2 Peter-Weyl-type theorem is shorter, which makes use of representation theory of compact groups.

Roughly speaking, the L^2 Peter-Weyl-type theorem claim that certain regular representation of the complex reductive group is completely reducible. We first give the precise definition.

Definition 4.1. Let G be a complex Lie group with a tame measure $d\mu$. The *right* and *left L^2 holomorphic regular representations* $\pi_{\mu,R}$ and $\pi_{\mu,L}$ of G on $\mathcal{H}L^2(G, d\mu)$ are defined by

$$(\pi_{\mu,R}(g)f)(h) = f(hg)$$

and

$$(\pi_{\mu,L}(g)f)(h) = f(g^{-1}h)$$

respectively, where $f \in \mathcal{H}L^2(G, d\mu)$, $g, h \in G$. For an element $g \in G$, the domains of $\pi_{\mu,R}(g)$ and $\pi_{\mu,L}(g)$ are

$$\mathcal{D}(\pi_{\mu,R}(g)) = \{f \in \mathcal{H}L^2(G, d\mu) : \int_G |f(hg)|^2\, d\mu(h) < +\infty\}$$

and

$$\mathcal{D}(\pi_{\mu,L}(g)) = \{f \in \mathcal{H}L^2(G, d\mu) : \int_G |f(g^{-1}h)|^2\, d\mu(h) < +\infty\}$$

respectively.

Remark 4.1. One should notice here that $\pi_{\mu,R}(g)$ and $\pi_{\mu,L}(g)$ are unbounded operators in general, that is, their domains are not necessarily the whole space $\mathcal{H}L^2(G, d\mu)$. But if some $g \in G$ such that $(r_g)_* d\mu = d\mu$ (or $(l_g)_* d\mu = d\mu$), then $\mathcal{D}(\pi_{\mu,R}(g))$ (or $\mathcal{D}(\pi_{\mu,L}(g))$) is the whole $\mathcal{H}L^2(G, d\mu)$. If G is complex reductive and $d\mu$ is K-admissible, we will show that there is a dense subspace $\mathcal{E}$ of $\mathcal{H}L^2(G, d\mu)$ such that $\mathcal{E} \subset \mathcal{D}(\pi_{\mu,R}(g)) \cap \mathcal{D}(\pi_{\mu,L}(g))$ for all $g \in G$.

Let G be a complex reductive group, K a maximal compact subgroup of it. For a finite-dimensional holomorphic representation π of G with representation space V_π of dimension d_π, we denote the linear span of the matrix elements $\{\pi_{ij} : i, j = 1, \cdots, d_\pi\}$ of π by $\mathcal{E}_\pi$, and let $\mathcal{E}$ be the linear span of $\{\mathcal{E}_\pi : [\pi] \in \widehat{G}_{\mathcal{H}}\}$. Note that for a tame measure μ on G, $C_{\pi,\mu} = \int_G \|\pi(g)\|^2\, d\mu(g) < +\infty$ if and only if $\mathcal{E}_\pi \subset \mathcal{H}L^2(G, d\mu)$. So if $d\mu$ is K-admissible, then $\mathcal{E} \subset \mathcal{H}L^2(G, d\mu)$. We let

$\widetilde{\pi}(g) = \pi(g^{-1})^t$, where A^t denotes the transpose of a matrix A. Then $\widetilde{\pi}$ is also a holomorphic representation of G, and $\widetilde{\pi}$ is irreducible if and only if π is irreducible.

Theorem 4.1. *Let G be a complex reductive group with a maximal compact subgroup K and a K-admissible measure $d\mu$. Then we have*
(i) $\mathcal{H}L^2(G, d\mu) = \bigoplus_{[\pi]\in\widehat{G}_{\mathcal{H}}} \mathcal{E}_\pi$, and the set

$$\{\frac{d_\pi}{\sqrt{C_{\pi,\mu}}}\pi_{ij} : i, j = 1, \cdots, d_\pi, [\pi] \in \widehat{G}_{\mathcal{H}}\}$$

is an orthonormal basis of $\mathcal{H}L^2(G, d\mu)$.
(ii) For each $g \in G$, the domains $\mathcal{D}(\pi_{\mu,R}(g))$ and $\mathcal{D}(\pi_{\mu,L}(g))$ contain the linear span $\mathcal{E}$ of $\{\mathcal{E}_\pi : [\pi] \in \widehat{G}_{\mathcal{H}}\}$, and hence are dense in $\mathcal{H}L^2(G, d\mu)$.
(iii) Both $\pi_{\mu,R}(g)$ and $\pi_{\mu,L}(g)$ are completely reducible and can be splited as direct sums of elements of $\widehat{G}_{\mathcal{H}}$, each $[\pi] \in \widehat{G}_{\mathcal{H}}$ occurs with multiplicity d_π.
(iv) For $1 \le i \le d_\pi$, the subspace of $\mathcal{E}_\pi$ (resp. $\mathcal{E}_{\widetilde{\pi}}$) spanned by the i-th row (resp. the i-th column) of the matrix (π_{ij}) (resp. $(\widetilde{\pi}_{ij})$) is invariant under $\pi_{\mu,R}$ (resp. $\pi_{\mu,L}$), and the restriction of $\pi_{\mu,R}$ (resp. $\pi_{\mu,L}$) to this subspace is equivalent to π.

Proof. We first show that for any $[\pi], [\pi'] \in \widehat{G}_{\mathcal{H}}$, we have

$$\int_G \pi_{ij}(g)\overline{\pi'_{kl}(g)}\, d\mu(g) = \begin{cases} \frac{C_{\pi,\mu}}{d_\pi^2}, & \text{if } [\pi] = [\pi'], i = k, j = l; \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

For each linear transform $A : V_{\pi'} \to V_\pi$, define

$$\overline{A} = \int_G \pi(g)A\pi'(g)^*\, d\mu(g),$$

where $\pi'(g)^*$ is the adjoint of $\pi'(g)$ with respect to the chosen inner product on $V_{\pi'}$. For each $x \in K$, by the K-left-invariance of $d\mu$, we can easily prove that $\pi(x)\overline{A}\pi'(x)^{-1} = \overline{A}$. So $\overline{A}$ is an intertwining operator between $\pi|_K$ and $\pi'|_K$. Since $\pi|_K$ and $\pi'|_K$ are irreducible, by Schur's Lemma, we have

$$\overline{A} = \int_G \pi(g)A\pi'(g)^*\, d\mu(g) = \begin{cases} 0, & [\pi] \ne [\pi']; \\ c_A I_{d_\pi}, & \pi = \pi', \end{cases}$$

where c_A is a constant. If $[\pi] \ne [\pi']$, by the arbitrariness of A, we have

$$\int_G \pi_{ij}(g)\overline{\pi'_{kl}(g)}\, d\mu(g) = 0.$$

That is, $\mathcal{E}_\pi \perp \mathcal{E}_{\pi'}$. Now assume $\pi = \pi'$. Then

$$\overline{A}_{ij} = \sum_{k,l} A_{kl} \int_G \pi_{ik}(g)\overline{\pi_{jl}(g)}\, d\mu(g) = c_A \delta_{ij},$$

where (A_{ij}) is the matrix form of A with respect to the chosen bases of V_π. If $i \ne j$, we have

$$\int_G \pi_{ik}(g)\overline{\pi_{jl}(g)}\, d\mu(g) = 0, \qquad i \ne j.$$

Hence two matrix elements of π which lie in different row are orthogonal. Now take $i = j$, $(A_{kl}) = E_{kk}$, where E_{kk} is the matrix with 1 at the (k, k)-position and 0 elsewhere, then we obtain

$$\int_G \pi_{ik}(g)\overline{\pi_{ik}(g)}\, d\mu(g) = c_{E_{kk}}$$

for $1 \le i \le d_\pi$. Hence the matrix elements lying in the same column have the same norm.

For an endomorphism A of V_π, we define

$$\overline{\overline{A}} = \int_G \pi(g)^* A\pi(g)\, d\mu(g).$$

Then, similarly, by the K-right-invariance of $d\mu$, $\overline{\overline{A}}$ is an intertwining operator of $\pi|_K$ and then $\overline{\overline{A}} = c'_A I_{d_\pi}$ for some constant c'_A. So we have

$$\overline{\overline{A}}_{ij} = \sum_{k,l} A_{kl} \int_G \pi_{lj}(g)\overline{\pi_{ki}(g)}\, d\mu(g) = c'_A \delta_{ij}.$$

Then

$$\int_G \pi_{lj}(g)\overline{\pi_{ki}(g)}\, d\mu(g) = 0, \qquad i \ne j,$$

$$\int_G \pi_{ki}(g)\overline{\pi_{ki}(g)}\, d\mu(g) = c'_{E_{kk}}.$$

Hence two matrix elements which lie in different columns are orthogonal and the matrix elements lying in the same row have the same norm.

Combining the above results, we get the conclusion that two different matrix elements of π are orthogonal and all matrix elements of π have the same norm. Moreover, we have

$$\begin{aligned}
&\int_G \pi_{ij}(g)\overline{\pi_{ij}(g)}\, d\mu(g) \\
=&\frac{1}{d_\pi^2} \int_G \sum_{k,l=1}^{d_\pi} \pi_{kl}(g)\overline{\pi_{kl}(g)}\, d\mu(g) \\
=&\frac{1}{d_\pi^2} \int_G \|\pi(g)\|^2\, d\mu(g) \\
=&\frac{C_{\pi,\mu}}{d_\pi^2}.
\end{aligned}$$

This completes the proof of (4.1).

Now for each $[\pi] \in \widehat{G}_{\mathcal{H}}$, consider the action of $\pi_{\mu,R}$ and $\pi_{\mu,L}$ on $\mathcal{E}_\pi$. We have

$$\begin{aligned}
(\pi_{\mu,R}(g)\pi_{ij})(h) =&\pi_{ij}(hg) = \sum_{k=1}^{d_\pi} \pi_{ik}(h)\pi_{kj}(g), \\
(\pi_{\mu,L}(g)\pi_{ij})(h) =&\pi_{ij}(g^{-1}h) = \sum_{k=1}^{d_\pi} \widetilde{\pi}_{ki}(g)\pi_{kj}(h).
\end{aligned}$$

That is,

$$\pi_{\mu,R}(g)\pi_{ij} = \sum_{k=1}^{d_\pi} \pi_{kj}(g)\pi_{ik},$$
$$\pi_{\mu,L}(g)\pi_{ij} = \sum_{k=1}^{d_\pi} \widetilde{\pi}_{ki}(g)\pi_{kj}.$$

Hence $\mathcal{E}_\pi$ is contained in the domains of $\pi_{\mu,R}(g)$ and $\pi_{\mu,L}(g)$, and for $1 \le i \le d_\pi$, the subspace of $\mathcal{E}_\pi$ spanned by the i-th row (resp. the i-th column) of the matrix (π_{ij}) is invariant under $\pi_{\mu,R}$ (resp. $\pi_{\mu,L}$), and the restriction of $\pi_{\mu,R}$ (resp. $\pi_{\mu,L}$) to this subspace is equivalent to π (resp. $\widetilde{\pi}$). For $\pi_{\mu,L}$, its restriction to the subspace of $\mathcal{E}_{\widetilde{\pi}}$ spanned by the i-th column of $(\widetilde{\pi}_{ij})$ is equivalent to $\widetilde{\widetilde{\pi}} = \pi$.

The set (4.2) spans a closed subspace $V = \overline{\mathcal{E}} = \bigoplus_{[\pi]\in\widehat{G}_{\mathcal{H}}} \mathcal{E}_\pi$ of $\mathcal{H}L^2(G, d\mu)$. We prove that $V = \mathcal{H}L^2(G, d\mu)$. If not, since V is invariant under $\pi_{\mu,R}$ and then invariant under $\pi_{\mu,R}|_K$, which is a unitary representation of K, $V^\perp \ne 0$ is also invariant under $\pi_{\mu,R}|_K$. Because a representation of a compact group is completely reducible (Folland [3], Theorem 5.2), we can choose an irreducible subspace W of $V^\perp$ under $\pi_{\mu,R}|_K$. By Proposition 2.3, the restriction of $\pi_{\mu,R}|_K$ on W is equivalent to $\pi|_K$ for some $[\pi] \in \widehat{G}_{\mathcal{H}}$. We choose a suitable basis $\{f_1, \cdots, f_{d_\pi}\}$ of W such that with respect to this basis, the matrix of the subrepresentation of $\pi_{\mu,R}|_K$ on W is $(\pi_{ij}|_K)$. Then

$$f_i(gx) = (\pi_{\mu,R}(x)f_i)(g) = \sum_{j=1}^{d_\pi} \pi_{ji}(x)f_j(g), \quad x \in K, g \in G, i = 1, \cdots, d_\pi.$$

Let $g = e$, we have

$$f_i(x) = \sum_{j=1}^{d_\pi} f_j(e)\pi_{ji}(x), \qquad x \in K.$$

Since f_i's and π_{ij}'s are all holomorphic on G, by Proposition 2.4, we have

$$f_i = \sum_{j=1}^{d_\pi} f_j(e)\pi_{ji}.$$

So $f_i \in \mathcal{E}_\pi$, which contradicts to $W \perp V$. The proof of the theorem is completed. □

Theorem 4.2. *Let G be a complex reductive group. Then $\mathcal{E}$ is a dense subspace of the Fréchet space $\mathcal{H}(G)$.*

Proof. Choose a maximal compact subgroup K of G. Let $f \in \mathcal{H}(G)$. By Theorem 3.2, there is a K-admissible measure $d\mu$ on G such that $f \in \mathcal{H}L^2(G, d\mu)$. By Theorem 4.1, f lies in the closure of $\mathcal{E}$ with respect to the Hilbert topology on $\mathcal{H}L^2(G, d\mu)$, hence lies in the Fréchet topology on $\mathcal{H}(G)$. This proves the theorem. □

5. Holomorphic Fourier transforms on complex reductive groups

As in the previous section, let G be a complex reductive group. Choose a maximal compact subgroup K of G. For each holomorphic function $f \in \mathcal{H}(G)$, by

Theorem 3.2, there exists a K-admissible measure $d\mu$ such that $f \in \mathcal{H}L^2(G, d\mu)$. Then by Theorem 4.1, we can expand f as

$$f = \sum_{[\pi]\in\widehat{G}_{\mathcal{H}}} \sum_{i,j=1}^{d_\pi} \lambda_{\pi,ij}\pi_{ij} \ , \tag{5.1}$$

where

$$\lambda_{\pi,ij} = \frac{d_\pi^2}{C_{\pi,\mu}} \int_G f(g)\overline{\pi_{ij}(g)}\, d\mu(g). \tag{5.2}$$

Equation (5.1) is called the *holomorphic Fourier series* of f. It obviously converges in $\mathcal{H}L^2(G, d\mu)$ by Theorem 4.1, hence also converges locally uniformly.

Remark 5.1. When $G = \mathbb{C}^*$, the Fourier series (5.1) is just the Laurant series in complex analysis.

Now we give the definition of the holomorphic Fourier transform of $f \in \mathcal{H}(G)$ as follows.

Definition 5.1. For $f \in \mathcal{H}(G)$, its *holomorphic Fourier transform* $\widehat{f}$ is defined by

$$\widehat{f}(\pi) = \frac{1}{C_{\pi,\mu}} \int_G f(g)\pi(g)^*\, d\mu(g), \qquad [\pi] \in \widehat{G}_{\mathcal{H}}, \tag{5.3}$$

where $d\mu$ is a K-admissible measure on G such that $f \in \mathcal{H}L^2(G, d\mu)$.

Note that $\widehat{f}(\pi)$ is an operator in the representation space V_π of π. With the orthonormal basis $\{e_j\}$ of V_π that we have chosen, $\widehat{f}(\pi)$ is given in the matrix form

$$\widehat{f}(\pi)_{ij} = \frac{1}{C_{\pi,\mu}} \int_G f(g)\overline{\pi_{ji}(g)}\, d\mu(g). \tag{5.4}$$

Comparing this equation with formula (5.2), we have

$$\widehat{f}(\pi)_{ij} = \frac{1}{d_\pi^2}\lambda_{\pi,ji}. \tag{5.5}$$

The following proposition shows that the Fourier transform is independent of the choice of K-admissible measures. To avoid the ambiguity, we denote the holomorphic Fourier transform of $f \in \mathcal{H}(G)$ by $\widehat{f}_\mu$ for the moment, when the K-admissible measure involved in the definition is $d\mu$.

Proposition 5.1. *Let $d\mu$ and $d\nu$ be two K-admissible measures on G. If $f \in \mathcal{H}L^2(G, d\mu) \bigcap \mathcal{H}L^2(G, d\nu)$, then $\widehat{f}_\mu = \widehat{f}_\nu$.*

Proof. To be precise, we rewrite the symbol $\lambda_{\pi,ij}$ defined in (5.2) by $\lambda^\mu_{\pi,ij}$. So by (5.1), we have

$$f = \sum_{[\pi]\in\widehat{G}_{\mathcal{H}}} \sum_{i,j=1}^{d_\pi} \lambda^\mu_{\pi,ij}\pi_{ij} \ .$$

Let K be a maximal compact subgroup with the normalized Haar measure dx. Since the above series converges locally uniformly, it converges uniformly on K. But on the compact group K, uniform convergence implies convergence in $L^2(K, dx)$, so the series

$$f|_K = \sum_{[\pi]\in\widehat{G}_{\mathcal{H}}} \sum_{i,j=1}^{d_\pi} \lambda^\mu_{\pi,ij}\pi_{ij}|_K$$

is in fact the Fourier series of $f|_K$ on K. Hence the coefficient $\lambda^{\mu}_{\pi,ij}$ is uniquely determined by $f|_K$, which is independent of μ. That is, $\lambda^{\mu}_{\pi,ij} = \lambda^{\nu}_{\pi,ij}$. By equation (5.5), we get $\widehat{f}_\mu = \widehat{f}_\nu$. □

Thanks to this proposition, we can omit the subscription and simply denote the Fourier transform of f by $\widehat{f}$.

Theorem 5.2. *For $f \in \mathcal{H}(G)$, we have the Fourier inversion formula*

$$f(g) = \sum_{[\pi]\in\widehat{G}_{\mathcal{H}}} d_\pi^2 \, tr(\widehat{f}(\pi)\pi(g)). \tag{5.6}$$

The series converges locally uniformly. If $d\mu$ is an K-admissible measure such that $f \in \mathcal{H}L^2(G, d\mu)$, then the series also converges in $\mathcal{H}L^2(G, d\mu)$.

Proof. Suppose $d\mu$ is a K-admissible measure with $f \in \mathcal{H}L^2(G, d\mu)$. By equation (5.5), we have

$$\begin{aligned}\sum_{i,j=1}^{d_\pi} \lambda_{\pi,ij}\pi_{ij}(g) =& d_\pi^2 \sum_{i,j=1}^{d_\pi} \widehat{f}(\pi)_{ji}\pi_{ij}(g)\\ =& d_\pi^2 \, tr(\widehat{f}(\pi)\pi(g)).\end{aligned}$$

By (5.1), we get

$$f(g) = \sum_{[\pi]\in\widehat{G}_{\mathcal{H}}} d_\pi^2 \, tr(\widehat{f}(\pi)\pi(g)).$$

It obviously converges in $\mathcal{H}L^2(G, d\mu)$, hence also converges locally uniformly. □

Corollary 5.3. *For $f \in \mathcal{H}(G)$, we have*

$$f(e) = \sum_{[\pi]\in\widehat{G}_{\mathcal{H}}} d_\pi^2 \, tr(\widehat{f}(\pi)). \tag{5.7}$$

□

Equation (5.7) can be viewed as a version of Plancherel Theorem, which reflects the completeness of the holomorphic Fourier analysis.

Now we give the relation between the Fourier transform of a function $f \in \mathcal{H}(G)$ and the Fourier transform of $R_g f$ and $L_g f$ for $g \in G$, where R_g and L_g are the right and left actions of g on $\mathcal{H}(G)$, respectively.

Theorem 5.4. *For $g \in G$ and $f \in \mathcal{H}(G)$, we have*

$$(R_g f)\widehat{\ }(\pi) = \pi(g)\widehat{f}(\pi). \tag{5.8}$$

$$(L_g f)\widehat{\ }(\pi) = \widehat{f}(\pi)\pi(g^{-1}). \tag{5.9}$$

Proof. We first prove (5.8). By Theorem 3.2, we can choose a K-admissible measure $d\mu$ on G such that $R_g f \in \mathcal{H}L^2(G, d\mu)$ for all $g \in G$. For $x \in K$, $\pi(x)$ is unitary, so

we have

$$\begin{aligned}(R_x f)\widehat{\ }(\pi) =& \frac{1}{C_{\pi,\mu}} \int_G f(hx)\pi(h)^* \, d\mu(h) \\ =& \frac{1}{C_{\pi,\mu}} \int_G f(h)\pi(hx^{-1})^* \, d\mu(h) \\ =& \frac{1}{C_{\pi,\mu}} \int_G f(h)\pi(x^{-1})^* \pi(h)^* \, d\mu(h) \\ =& \frac{\pi(x)}{C_{\pi,\mu}} \int_G f(h)\pi(h)^* \, d\mu(h) \\ =& \pi(x)\widehat{f}(\pi).\end{aligned}$$

That is, $(R_x f)\widehat{\ }(\pi) = \pi(x)\widehat{f}(\pi)$ for all $x \in K$. But $(R_g f)\widehat{\ }(\pi)$ and $\pi(g)\widehat{f}(\pi)$ are holomorphic with respect to $g \in G$, by Proposition 2.4, $(R_g f)\widehat{\ }(\pi) = \pi(g)\widehat{f}(\pi)$ for all $g \in G$. This proves (5.8). The proof of (5.9) is similar. □

We now consider the Fourier transform of a function under the action of invariant differential operators. Recall that a left-invariant differential operator D on G can be viewed as an element of the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G, and a finite-dimensional representation π of G induces a representation of $U(\mathfrak{g})$, which we also denote by π.

Corollary 5.5. *Suppose D is a left-invariant differential operator on G, then for $f \in \mathcal{H}(G)$, we have*

$$(Df)\widehat{\ }(\pi) = \pi(D)\widehat{f}(\pi). \tag{5.10}$$

Proof. By the Poincaré-Birkhoff-Witt Theorem, D can be expressed as a linear combination of the monomials $X_1 X_2 \cdots X_r$, where $X_i \in \mathfrak{g}$. So without loss of generality, we may assume $D = X \in \mathfrak{g}$. Now

$$\begin{aligned}(Xf)(g) =& \frac{d}{dt}\Big|_{t=0} f(ge^{tX}) \\ =& \frac{d}{dt}\Big|_{t=0} (R_{e^{tX}} f)(g).\end{aligned}$$

So by (5.8),

$$\begin{aligned}(Xf)\widehat{\ }(\pi) =& \frac{d}{dt}\Big|_{t=0} (R_{e^{tX}} f)\widehat{\ }(\pi) \\ =& \frac{d}{dt}\Big|_{t=0} \pi(e^{tX})\widehat{f}(\pi) \\ =& \pi(X)\widehat{f}(\pi).\end{aligned}$$

This completes the proof. □

For right-invariant differential operator we have a similar argument, which is deduced from (5.9). We state the conclusion below. Its proof is similar to that of the above corollary, we omit it here.

Corollary 5.6. *Suppose D' is a right-invariant differential operator on G, then for $f \in \mathcal{H}(G)$, we have*

$$(D'f)\widehat{\ }(\pi) = \widehat{f}(\pi)\pi(D'). \tag{5.11}$$

□

6. Applications of the Holomorphic Fourier transform

In this section, we give two applications of the holomorphic Fourier transform. One is to holomorphic class functions on complex reductive groups, the other is to holomorphic evolution PDEs on such groups.

First we prove that the linear span $\mathcal{E}_c$ of irreducible holomorphic characters of a complex reductive group G is dense in the Fréchet space $\mathcal{H}_c(G)$ of holomorphic class functions on G. We prove this by showing that the irreducible holomorphic characters of a G form an orthogonal basis of the subspace of class functions in $\mathcal{H}L^2(G, d\mu)$, where $d\mu$ is a K-admissible measure on G. For a finite-dimensional holomorphic representation π of G, the character χ_π of π is defined by $\chi_\pi(g) = \operatorname{tr}\pi(g)$. A function $f \in \mathcal{H}(G)$ is called a class function if $f(gh) = f(hg)$ for all $g, h \in G$. It is obvious that holomorphic characters are class functions. We denote the linear span of irreducible holomorphic characters by $\mathcal{E}_c$, the Fréchet space of class holomorphic functions by $\mathcal{H}_c(G)$, and the subspace of class functions in $\mathcal{H}L^2(G, d\mu)$ by $\mathcal{H}L^2_c(G, d\mu)$.

Theorem 6.1. *(i)* $\{\sqrt{\frac{d_\pi}{C_{\pi,\mu}}}\,\chi_\pi : [\pi] \in \widehat{G}_{\mathcal{H}}\}$ *is an orthonormal basis of* $\mathcal{H}L^2_c(G, d\mu)$.
(ii) $\mathcal{E}_c$ *is dense in* $\mathcal{H}_c(G)$ *with respect to the Fréchet topology.*

Proof. By Theorem 4.1, it is obvious that $\{\sqrt{\frac{d_\pi}{C_{\pi,\mu}}}\,\chi_\pi : [\pi] \in \widehat{G}_{\mathcal{H}}\}$ is an orthonormal set in $\mathcal{H}L^2_c(G, d\mu)$. To prove the completeness, it is enough to show that each $f \in \mathcal{H}L^2_c(G, d\mu)$ can be expressed as a series of the form $\sum_{[\pi]\in\widehat{G}_{\mathcal{H}}} a_\pi \chi_\pi$. Notice that for each $[\pi] \in \widehat{G}_{\mathcal{H}}$, $\pi|_K$ is unitary, So for $x \in K$, we have

$$\begin{aligned}
\pi(x)\widehat{f}(\pi)\pi(x)^{-1} =& \frac{1}{C_{\pi,\mu}} \int_G f(g)\pi(x^{-1})^*\pi(g)^*\pi(x)^*\, d\mu(g) \\
=& \frac{1}{C_{\pi,\mu}} \int_G f(g)\pi(xgx^{-1})^*\, d\mu(g) \\
=& \frac{1}{C_{\pi,\mu}} \int_G f(x^{-1}gx)\pi(g)^*\, d\mu(g) \\
=& \frac{1}{C_{\pi,\mu}} \int_G f(g)\pi(g)^*\, d\mu(g) \\
=& \widehat{f}(\pi).
\end{aligned}$$

This shows $\pi(g)\widehat{f}(\pi)$ and $\widehat{f}(\pi)\pi(g)$ agree on K. But they are holomorphic on G, by Proposition 2.4, $\pi(g)\widehat{f}(\pi) = \widehat{f}(\pi)\pi(g)$ for all $g \in G$. Since π is irreducible, by Schur's Lemma, $\widehat{f}(\pi) = b_\pi I$ for some constant b_π. By the Fourier inversion formula (5.6), we have $f = \sum_{[\pi]\in\widehat{G}_{\mathcal{H}}} d_\pi^2 b_\pi \chi_\pi$, which converges both in $\mathcal{H}L^2_c(G, d\mu)$ and locally uniformly. This proves the theorem. □

Corollary 6.2. *Every* $f \in \mathcal{H}_c(G)$ *can be expanded uniquely as*

$$f = \sum_{[\pi]\in\widehat{G}_{\mathcal{H}}} a_\pi \chi_\pi.$$

The series converges locally uniformly. The coefficients a_π can be expressed as

$$a_\pi = \frac{d_\pi}{C_{\pi,\mu}} \int_G f(g)\overline{\chi_\pi(g)}\, d\mu(g)$$

for each K-admissible measure $d\mu$ on G such that $f \in \mathcal{H}L^2(G, d\mu)$.

Proof. Choose a K-admissible measure $d\mu$ such that $f \in \mathcal{H}L^2(G, d\mu)$ and apply Theorem 6.1. □

Our next application of the holomorphic Fourier transform concerns holomorphic evolution partial differential equations on a complex reductive group G. Suppose D is a left-invariant differential operator on G. Consider the evolution equation with holomorphic initial condition

$$\begin{cases} \frac{d}{dt} f_t = D f_t \\ f_0 \in \mathcal{H}(G) \end{cases} \tag{6.1}$$

on G. We have the following result.

Theorem 6.3. *If equation (6.1) has a holomorphic solution $f_t \in \mathcal{H}(G)$ for $t > 0$, then the solution f_t can be expressed as*

$$f_t(g) = \sum_{[\pi]\in \widehat{G}_{\mathcal{H}}} d_\pi^2\, tr(e^{t\pi(D)} \widehat{f}_0(\pi)\pi(g)). \tag{6.2}$$

The series converges locally uniformly. Conversely, if the series in the right hand side of (6.2) converges locally uniformly, then it converges to a holomorphic solution of equation (6.1).

Proof. Suppose equation (6.1) has a holomorphic solution $f_t \in \mathcal{H}(G)$ for $t > 0$. Applying the holomorphic Fourier transform to both side of the equation, by Corollary 5.5, we get

$$\frac{d}{dt}\widehat{f}_t(\pi) = (Df_t)\widehat{\ }(\pi) = \pi(D)\widehat{f}_t(\pi).$$

This ordinary differential equation of matrix form can be solved directly as

$$\widehat{f}_t(\pi) = e^{t\pi(D)}\widehat{f}_0(\pi).$$

By the Fourier inversion formula (5.6), we get the desired expression of f_t. Conversely, if the series in the right hand side of (6.2) converges locally uniformly, then a straightforward computation shows that the resulting function f_t determined by (6.2) dose form a holomorphic solution of equation (6.1). □

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